## Silence is the only homogeneous sound field in unbounded space

- Sound field with no boundaries and no incoming field
  - 3- d wave equation which satisfies the radiation condition is

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\frac{f(t-r/c)}{r}=0$$

  - With the closer inspection at the point of *r=0* , the wave equation is reduce to the Laplace equation because spatial derivatives are more singular

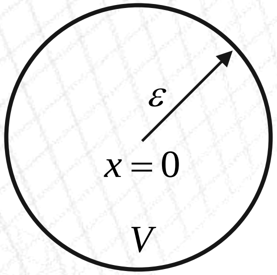
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\frac{f(t-r/c)}{r}\rightarrow-\nabla^2\,\frac{f(t-r/c)}{r}$$

Since $\nabla^2$ term is more singular

# Sources of Sound

## ❖ Silence is the only homogeneous sound field in unbounded space

- Consider the small volume surrounding the origin is finite



surface $S$ of area $4\pi\varepsilon^2$

$$\int_V \nabla^2 \frac{f\left(t-r/c\right)}{r} = \int_s \frac{\partial}{\partial n}\left(\frac{f\left(t-r/c\right)}{r}\right) dS$$

$$= 4\pi\varepsilon^2 \left\{ -\frac{f\left(t-\varepsilon/c\right)}{\varepsilon^2} + \frac{1}{\varepsilon}\frac{\partial f}{\partial r}\left(t-\varepsilon/c\right) \right\}$$

$$= -4\pi f\left(t\right) \text{ as } \varepsilon \to 0$$

- 3-D delta function $\delta(x)$

$$\int_V \delta(x) dV = 1 \qquad \textit{If V includes x = 0}$$

$$= 0 \qquad \textit{otherwise}$$

# Sources of Sound

## Silence is the only homogeneous sound field in unbounded space

- Both sides of equation is set to be zero on the homogeneous wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\frac{f(t-r/c)}{r}=4\pi f(t)\delta(\mathbf{x}) \qquad - ①$$

only homogeneous condition of - ① , $f=0$

- Generally, eq. ① can be written as

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2}-\nabla^2 p'=q(\vec{x},t)$$

In sound field, $q = 0$
In source region, $q \neq 0$

# Sources of Sound

## ❖ The definition of a sound source

- The sound with very weak disturbances
  - Three dimensional wave equation is

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = q$$

  - In the sound field, *q* must be zero. In the source region, *q* is non-zero

$$\frac{1}{c^2}\frac{\partial^2\left(p' + f(q)\right)}{\partial t^2} - \nabla^2\left(p' + f(q)\right) = q + \frac{1}{c^2}\frac{\partial^2 f(q)}{\partial t^2} - \nabla^2 f(q)$$

Sound field : $p' + f(q) = p'$

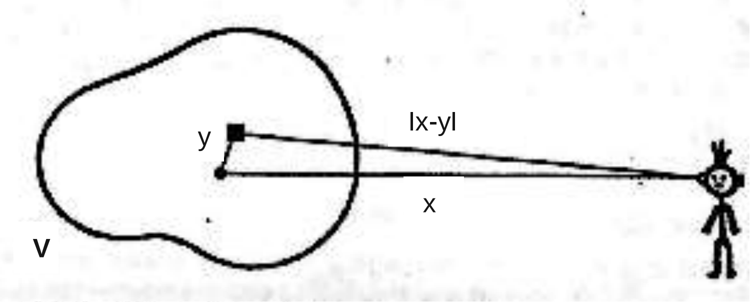
Source field : $q + \frac{1}{c^2}\frac{\partial^2 f(q)}{\partial t^2} - \nabla^2 f(q)$

  - The wave field does not contain sufficient information for its sources to be identified. But the sound field generated by a source field *q* is unique.

# Sources of Sound

## The definition of a sound source

- Retarded time
  - The sound arriving at ***x*** at time *t* must have been launched from the source at the previous time $\tau = t-|\mathbf{x}-\mathbf{y}|/c$, which is usually called the retarded time.



$$q(\mathbf{x},t) = \int_V q(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y})d^3\mathbf{y}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\frac{q(\mathbf{y},t-r/c)}{r} = 4\pi q(\mathbf{y},t)\delta(\mathbf{x}-\mathbf{y})$$

$$p'(\vec{\mathbf{x}},t) = \int_V \frac{q(\vec{\mathbf{y}},t-|\vec{\mathbf{x}}-\vec{\mathbf{y}}|/c)}{4\pi|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}d^3\vec{\mathbf{y}}$$

  - If the source region(*V*) is compact, the source region is negligible for a distant observer( $|\mathrm{x}| \gg |\mathrm{y}|$ ).

$$p'(\vec{\mathbf{x}},t) = \frac{q(t-|\vec{\mathbf{x}}|/c)}{4\pi|\vec{\mathbf{x}}|} \qquad \frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = -\mathrm{div}f(\mathbf{x},t) = -\frac{\partial}{\partial x_i}f_i(\mathbf{x},t)$$

# ❖ The definition of a sound source

## Monopole and dipole source distribution

- The monopole source distribution q(x,t) is said to be degenerate into a dipole distribution

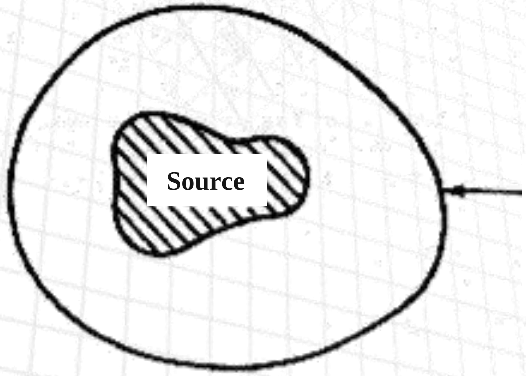
$$q(\vec{\mathbf{x}},t) = -\nabla\cdot f(\vec{x},t)$$

$q(\vec{\mathbf{x}},t)$ : monopole distribution

$f(\vec{x},t)$ : dipole distribution

- After differentiation with respect to the $x_i$,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\frac{\partial}{\partial x_i}\int_V \frac{f_i\left(\vec{\mathbf{y}},t-|\vec{\mathbf{x}}-\vec{\mathbf{y}}|/c\right)}{4\pi|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}d^3y = \frac{\partial f_i(\vec{x},t)}{\partial x_i}$$



**Surface *S* encloses volume *V* in which lies all the region where f≠0**

So, $p'(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\int_V \frac{f_i\left(\vec{\mathbf{y}},t-|\vec{\mathbf{x}}-\vec{\mathbf{y}}|/c\right)}{4\pi|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}d^3\mathbf{y}$

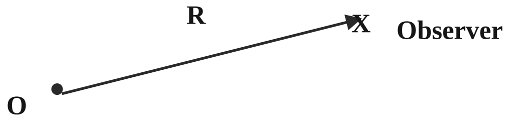
Equivalent monopole formula,

$$p'(\mathbf{x},t) = -\int_V \frac{\nabla\cdot\vec{f}\left(\vec{\mathbf{y}},t-|\vec{\mathbf{x}}-\vec{\mathbf{y}}|/c\right)}{4\pi|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}d^3\mathbf{y}$$

# Sources of Sound

## ❖ The definition of a sound source

- Point monopole and dipoles
  - 1) point monopole



$$q(\vec{x},t)=Q(t)\,f(\vec{x})$$

Source strength

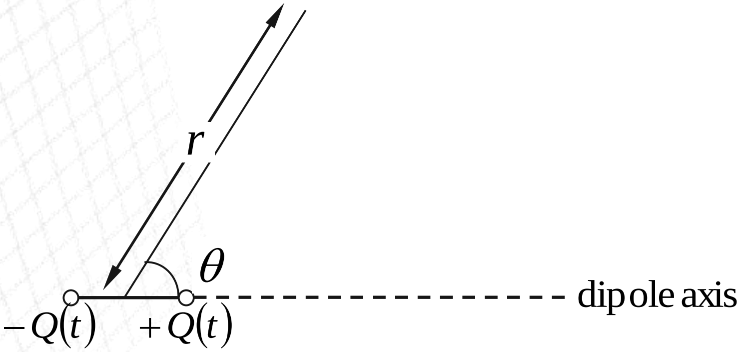
- Source : $p'(\vec{\mathbf{x}},t)=\int_V \frac{q(\vec{\mathbf{y}},t-|\vec{\mathbf{x}}-\vec{\mathbf{y}}|/c)}{4\pi|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}d^3\vec{\mathbf{y}}$
- For compact source $p'(\mathbf{x},t)=\frac{Q(t-r/c)}{4\pi r}$ **omni-directional**

  - 2) Point Dipole

$$q(\vec{x},t)=-\nabla\cdot(\vec{F}(t)\delta(\vec{x}))=-\tfrac{\partial}{\partial x_i}(F_i(t)\delta(\vec{x}))$$

Dipole strength with direction

# ❖ The definition of a sound source



- Combined source strength

$$-Q(t)\{\delta(\mathbf{x})-\delta(\mathbf{x}-\boldsymbol{\delta}\mathbf{l}\}$$

$$=-Q(t)\left\{\delta l_1\frac{\partial}{\partial x_1}\delta(\mathbf{x})+\delta l_2\frac{\partial}{\partial x_2}\delta(\mathbf{x})+\delta l_3\frac{\partial}{\partial x_3}\delta(\mathbf{x})\right\}\quad \text{as}\quad \vec{\delta l}\to 0$$

$$=-\mathrm{div}\{Q(t)\delta l\,\delta(\mathbf{x})\}\quad \text{as}\quad \vec{\delta l}\to 0$$

- *Source :* $\overrightarrow{F(t)}=Q(t)\overrightarrow{\delta l}$

$$p'(\mathbf{x},t)=-\frac{\partial}{\partial x_i}\int_V\frac{f_i(\vec{\mathbf{y}},t-|\vec{\mathbf{x}}-\vec{\mathbf{y}}|/c)}{4\pi|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}d^3\vec{\mathbf{y}}\qquad p'_d(\mathbf{x},t)=\frac{\partial}{\partial x_i}\left[\frac{F_i(t-r/c)}{4\pi r}\right]\qquad p'_d(r,\theta,t)=\frac{\cos\theta}{4\pi}\left\{\frac{F}{r^2}+\frac{1}{cr}\frac{\partial F}{\partial t}\right\}$$

## The definition of a sound source

- Note
  - *i)* the sound field of dipole source has an angular dependence
  - *ii)* no disturbance at 90° to the dipole axis(their sound exactly cancel)
  - *iii)* Pressure disturbance have a maximum magnitude on the dipole axis
  - *iv)* $P`_{dipole}$ = Near-field term ($\sim 1/r^2$)+ Far-field term ($\sim 1/r$)

    $P`_{monopole}$ = Far-field term only ($\sim 1/r$)
  - *v)* Dipole has some cancellation of the acoustic field,

    so dipoles are less efficient radiations than monopoles. (by $M^2$)

# The definition of a sound source

- (Example)
  - Two point sources ; (1) monopole, (2) dipole at r=0
  - radiates sound of frequency $\omega$.
  - Determine the ratio of maximum pressure disturbance at $r=\varepsilon(\ll c/\omega)$ to that at $r=l\gg c/\omega$) for the two sources.

  - *i*) pressure perturbation of monopole

$$p'(r,t)=\frac{A}{4\pi r}e^{i\omega(t-r/c)}$$

$$\left.p'_{\max}\right]_{r=\varepsilon}=\frac{A}{4\pi\varepsilon}$$

$$\left.p'_{\max}\right]_{r=l}=\frac{A}{4\pi l}$$

$$Ratio=\frac{l}{\varepsilon}$$

## ❖ The definition of a sound source

- *ii*) pressure perturbation of dipole

$$p'(r,t) = \frac{\cos\theta}{4\pi} B\left\{\frac{i\omega}{cr} + \frac{1}{r^2}\right\} e^{i\omega(t - r/c)}$$

$$\left. p'_{\max} \right]_{r=\varepsilon} = \frac{B}{4\pi\varepsilon^2}$$

$$\left. p'_{\max} \right]_{r=l} = \frac{B}{4\pi lc}$$

$$Ratio = \frac{l}{\varepsilon}\left(\frac{c}{\omega\varepsilon}\right)$$

- $\varepsilon\omega/c \ll 1$, so $\frac{l}{\varepsilon}\left(\frac{c}{\omega\varepsilon}\right) \gg \frac{l}{\varepsilon}$

- Near-field disturbance decays very fast, so does not does not radiate far away as sound.

# Sources of Sound

## Acoustic source process

- What mechanisms do produce these source distribution in nature?

  - Source process $\equiv$ one in which there is a forcing term in the wave equation
    - Violate the homogeneous wave equation!!

  - Common sources
    - Vibration of the surface
    - Breakdown of the state equation $p=f(\rho)$

      (unsteady heat addition to fluid)

# Sources of Sound

## ❖ Acoustic source process

### ● Combustion noise

- When heat is added to a fluid the density of the fluid a function of two variables of pressure and heat   *ρ=ρ(p,h)*

- Hence, $$d\rho = \left.\frac{\partial \rho}{\partial p}\right|_h dp + \left.\frac{\partial \rho}{\partial h}\right|_p dh = \frac{1}{c^2} dp + \left.\frac{\partial \rho}{\partial h}\right|_p dh$$

- For a perfect gas, $$\rho = p/RT \rightarrow \left.\frac{\partial \rho}{\partial h}\right|_p = -\frac{p}{RT^2}\left.\frac{\partial T}{\partial h}\right|_p = \frac{p}{RT^2 c_p} = -\frac{\rho_0(\gamma-1)}{c^2}$$

- Finally, $$d\rho = \frac{1}{c^2} dp - \frac{\rho_0(\gamma-1)}{c^2} dh$$

- The linearized equation describing motion of a perfect gas with unsteady heat is

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \frac{\rho_0(\gamma-1)}{c^2}\frac{\partial^2 h}{\partial t^2}$$

## ❖ Acoustic source process

- The solution of wave equation is

$$p'(\mathbf{x},t)=\frac{\rho_0(\gamma-1)}{4\pi c^2}\frac{\partial^2}{\partial t^2}\int\frac{h(\vec{\mathbf{y}},t-|\vec{\mathbf{x}}-\vec{\mathbf{y}}|/c)}{|\vec{\mathbf{x}}-\vec{\mathbf{y}}|}d^3\vec{\mathbf{y}}$$

- The sound field is forced by the rate of change in the heat addition rate. Steady heating is silent

- Actually, the sound of the monopole source caused by unsteady heating is purely due to the unsteady expansion of the fluid at essentially constant pressure.

$$\frac{\rho_0(\gamma-1)}{c^2}\frac{\partial^2 h}{\partial t^2}=-\left.\frac{\partial^2\rho}{\partial t^2}\right|_p$$

- ∴ Monopole strength = $\rho_0=\frac{\partial}{\partial t}(\nabla\cdot\vec{V})$

Rate of change in the fractional rate of volumetric growth

# Sources of Sound

## ❖ Acoustic source process

- Sound generation by the linear creation of matter and externally applied forces
  - The acoustic sources by the mass creation

$$\frac{\partial \rho}{\partial t} + \rho_0 \mathrm{div}\ \mathbf{v} = m$$

  - The linearized momentum equation with applied forces

$$\rho_0 \frac{\partial \mathbf{v}}{\partial} + \mathrm{grad}\ \ p' = \mathbf{f}$$

  - The density would be function of a pressure and the newly created mass. Suppose that total density volume is consisted of created mass and rest of the fluid $\rho = \beta\rho_m + (1-\beta)\rho_f$ , $m = \frac{\partial}{\partial t}(\beta\rho_m)$

# Sources of Sound

## ❖ Acoustic source process

- The density fluctuation with respect to time is

$$\frac{\partial \rho}{\partial t} = \beta \frac{\partial \rho_m}{\partial t} + (1-\beta)\frac{\partial \rho_f}{\partial t} = m + \frac{1}{c^2}\frac{\partial p'}{\partial t} - \frac{\partial}{\partial t}(\beta \rho_0)$$

- So, $\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial t^2}(\beta \rho_0) - \frac{\partial m}{\partial t}$

- The wave equation above these relation is

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \underbrace{\rho_0 \frac{\partial^2 \beta}{\partial t^2}}_{\text{Acceleration term}} - \underbrace{\operatorname{div} \vec{\mathbf{f}}}_{\text{External Forcing term}}$$

- It is only the volume occupied by that newly created mass that drives sound field

# Sources of Sound

## ❖ Acoustic source process

- Note
  - 1) steady creation of mass is silent

    (only accelerating growth makes sound)
  - 2) Monopole strength $\rho_0 \frac{\partial^2 p}{\partial t^2} \Big/ unit\ volume$
  - 3) dipole strength $-\nabla \cdot \vec{f}$

- The solution of wave equation is

$$p'(\mathbf{x},t) = \rho_0 \frac{\partial^2}{\partial t^2} \int_V \frac{\beta(\mathbf{y}, t - |\mathbf{x}-\mathbf{y}|/c)}{4\pi|\mathbf{x}-\mathbf{y}|} d^3\mathbf{y} - \frac{\partial}{\partial x_i} \int_V \frac{f_i(\mathbf{y}, t - |\mathbf{x}-\mathbf{y}|/c)}{4\pi|\mathbf{x}-\mathbf{y}|} d^3\mathbf{y}$$

  - If the flow field has non-linear effects, sound waves are also generated.(quadrupole)

## Sources of Sound

### ❖ Lighthill's acoustic analogy

- Sound generation by flow
  - Sound – " ***Weak motion about a homogenous state of rest*** "
  - Source of sound by Lighthill (1951)

  " ***The difference between the exact statement of natural laws and their acoustical approximations*** "
  - The exact statement mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

  - The exact statement of momentum conservation

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(p_{ij} + \rho v_i v_j) = 0 \qquad (p_{ij} = p'\delta_{ij} - \tau_{ij}\delta_{ij})$$

  - The manipulation of above two equations

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2 \rho'}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}(p_{ij} + \rho v_i v_j)$$

# Sources of Sound

## ❖ Lighthill’s acoustic analogy

- Subtract density 2nd order derivatives and Lighthill’s equation is derived

$$\frac{\partial^2 \rho'}{\partial t^2} - c^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \qquad \left(T_{ij} = \rho v_i v_j + p_{ij} - c^2 \rho' \delta_{ij}\right)$$

- $T_{ij}$ is the source and is called “ Lighthill stress tensor ”

- The sound field by the exact equations is

$$\rho'(\mathbf{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_V \frac{T_{ij}\left(\mathbf{y}, t - |\mathbf{x} - \mathbf{y}|/c\right)}{4\pi c^2 |\mathbf{x} - \mathbf{y}|} dV$$

- $T_{ij}$ is of course zero in the sound field if we have linear motion only
- However, $T_{ij}$ cannot be vanished in the turbulent flows

# Sources of Sound

## ❖ Lighthill's acoustic analogy

- Note
  - *i*) No approximation made so far
  - *ii*) Real material motion ↔ acoustic motion

Quadrupole ↔ { Monopole, Dopole }

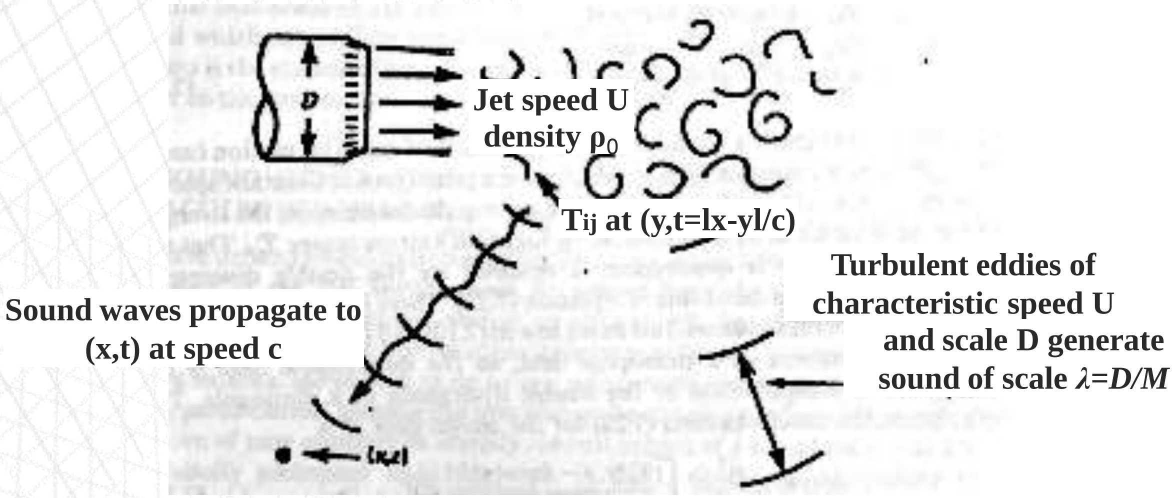
Non- linear — Linear

  - *iii*) The strength of the quadrupole is $T_{ij}$
  - *iv*) Double divergence - double tendency for source elements to cancel
    - Dipole field - the divergence of a monopole field
    - Quadrupole field - the divergence of a dipole field
      - the double divergence of a monopole field

# Sources of Sound

## ❖ Lighthill's acoustic analogy

- Jet Noise
  - Consider a jet flow of velocity *U* and diameter *D*



**Jet speed U**
**density $\rho_0$**

**$T_{ij}$ at (y,t=lx-yl/c)**

**Turbulent eddies of characteristic speed U and scale D generate sound of scale $\lambda$=*D/M***

**Sound waves propagate to (x,t) at speed c**

  - Characteristic frequency and wavelength is defined by dimensional coefficient of jet of flow and diameter

$$\lambda = c\frac{D}{U} = DM^{-1}$$

# Sources of Sound

## ❖ Lighthill's acoustic analogy

- For low Mach number ($\lambda \gg D$) , the source region is compact.
  → the retarded time can be negligible
- The sound field is dimensionalized by belows :

$$\frac{\partial}{\partial x_i} \longrightarrow \frac{1}{\lambda} \text{ or } \frac{M}{D}$$

$$\int dV \longrightarrow D^3$$

$$T_{ij} \longrightarrow \rho_0 U^2$$

- Together these give the estimate for low Mach number

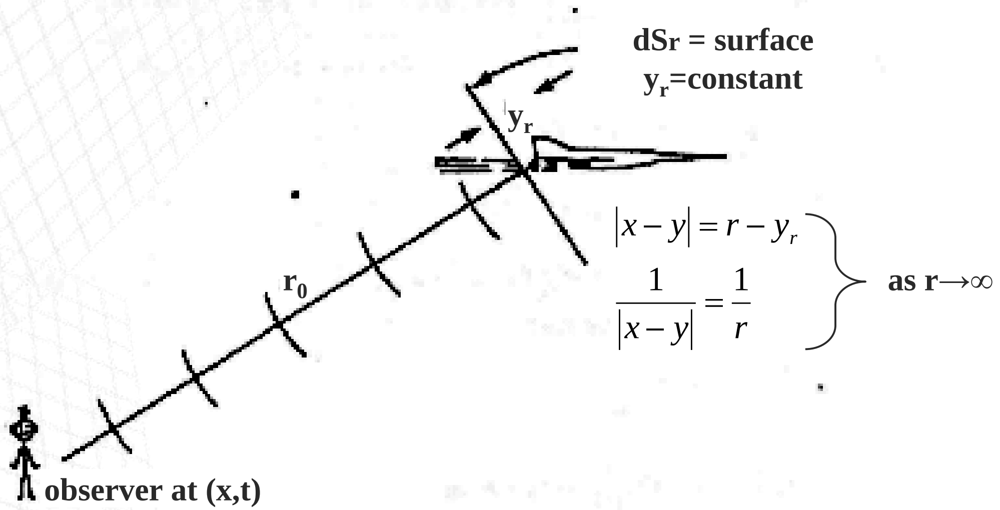
$$\rho' \approx \rho_0 M^4 \frac{D}{|\mathbf{x}|} \qquad \longrightarrow \qquad \overline{\rho'^2} \approx \rho_0^2 M^8 \frac{D^2}{|\mathbf{x}|^2} \qquad \text{when M<<1}$$

"Eighth power low"

- Low Mach number jet are acoustically very inefficient, as are all compact sources

# Lighthill’s acoustic analogy

- For high Mach number, supersonic jets ($\lambda \ll D$),
  - the sound field is dominated by retarded time
  - Far-field approximation



$$|x-y| = r - y_r$$

$$\frac{1}{|x-y|} = \frac{1}{r}$$

$$|\mathbf{x}-\mathbf{y}| = |\mathbf{x}| - \frac{\mathbf{y}\cdot\mathbf{x}}{|\mathbf{x}|} + O\left(\frac{1}{|\mathbf{x}|^2}\right)$$

# Sources of Sound

## ❖ Lighthill’s acoustic analogy

- The sound field with far-field approximation is

$$\rho' = \frac{1}{4\pi c^2}\frac{\partial^2}{\partial r^2}\int T_{rr}(\mathbf{y},\tau)dV \qquad \left(T_{rr} = \left(x_i x_j \big/ |x|^2\right)T_{ij}\right)$$

- In order to bring out the retarded time variations,

$$\rho' = \frac{1}{4\pi cr}\frac{\partial^2}{\partial r^2}\int T_{rr}(\mathbf{y},\tau)dS_r d\tau \qquad \mathbf{y} = (y_s, y_r) = (y_s, c(\tau - t) + r)$$

- The characteristic magnitude of the density perturbation in the sound induced by supersonic jets is estimated by setting

$$\frac{\partial}{\partial r} \longrightarrow \frac{1}{D} \qquad T_{ij} \longrightarrow \rho_0 U^2$$

$$\int dS \longrightarrow D^2 \qquad \int d\tau \longrightarrow \frac{D}{U}$$

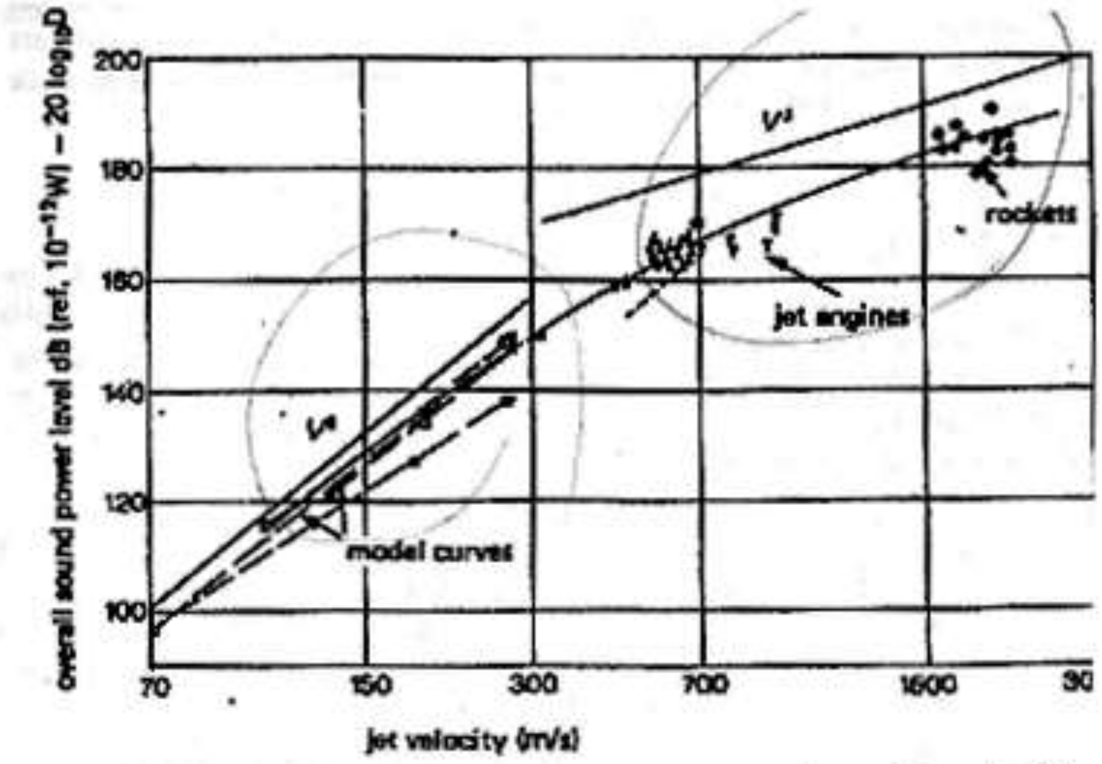
- Density and the mean square noise level is

$$\rho' \approx \rho_0 M \frac{D}{r} \qquad \longrightarrow \qquad \overline{\rho}'^2 \approx \rho_0^2 M^2 \frac{D^2}{r^2}$$

# Sources of Sound

## ❖ Lighthill’s acoustic analogy

- Actually, there is one other effect to do with the fact that the number of eddies that can be heard at any time increases with Mach number which changes this law to the velocity cubed dependence that is in agreement with the experimental observation



# Sources of Sound

## ❖ The sound from the stationary foreign bodies

- The foreign bodies in the flow

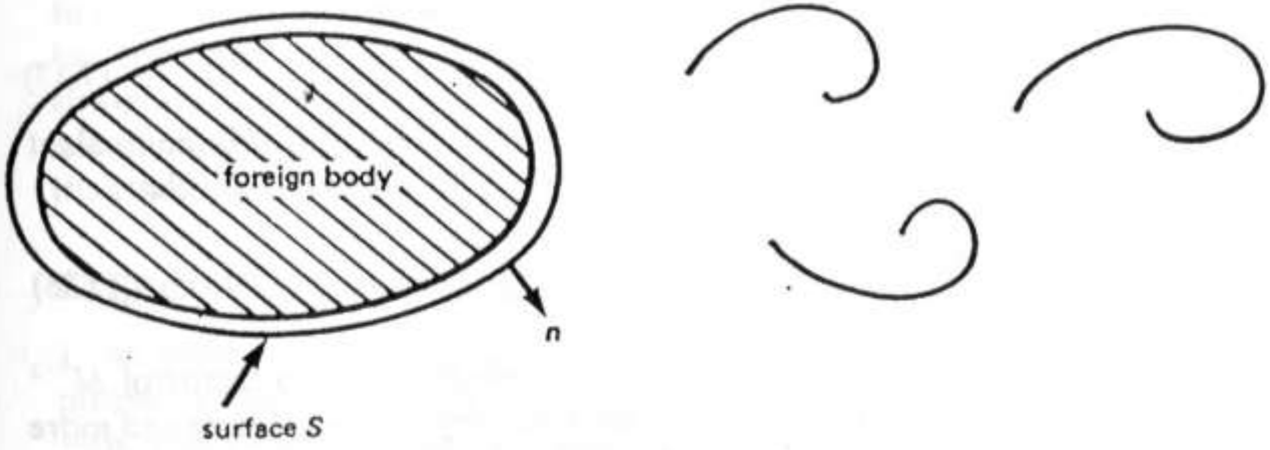


Fig. 7.8 — A stationary control surface enclosing a foreign body.

- Evaluate $$\rho'(\mathbf{x},t)=\frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{T_{ij}\left(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c\right)}{4\pi c^2|\mathbf{x}-\mathbf{y}|}dV$$

  Through some mathematical procedures, (pp. 163-164, text)

# Sources of Sound

## ❖ The sound from the stationary foreign bodies

- When there are foreign bodies in the flow it is sometimes convenient to lump together the quadrupoles distributed over the interior of foreign bodies.
- First, the derivatives of x with respect to the observer position are rewritten in terms of the derivatives of y with respect to the source position

$$\frac{\partial}{\partial x_i}\left\{\frac{T_{ij}\left(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c\right)}{|\mathbf{x}-\mathbf{y}|}\right\}=\frac{\partial T_{ij}/\partial y_i}{|\mathbf{x}-\mathbf{y}|}-\frac{\partial}{\partial y_i}\left\{\frac{T_{ij}\left(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c\right)}{|\mathbf{x}-\mathbf{y}|}\right\}$$

- $T_{ij}$ can be re-arranged as,([] implies that function of retarded time)

$$\frac{\partial}{\partial x_i}\left[\frac{T_{ij}}{r}\right]=-\frac{\partial}{\partial y_i}\left[\frac{\rho v_i v_j}{r}\right]-c^2\frac{\partial}{\partial x_j}\left[\frac{p'}{r}\right]-\left[\frac{1}{r}\frac{\partial}{\partial t}\left(\rho v_j\right)\right]$$

- Differentiate with respect to the $x_j$ gives

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{T_{ij}}{r}\right]=-\frac{\partial^2}{\partial y_i \partial x_j}\left[\frac{\rho v_i v_j}{r}\right]-c^2\frac{\partial^2}{\partial x_j \partial x_j}\left[\frac{p'}{r}\right]-\frac{\partial}{\partial x_j}\left[\frac{1}{r}\frac{\partial}{\partial t}\left(\rho v_j\right)\right]$$

# Sources of Sound

## ❖ The sound from the stationary foreign bodies

- The procedure described above shows that

$$\frac{\partial}{\partial x_j}\left[\frac{1}{r}\frac{\partial}{\partial t}\left(\rho v_j\right)\right]=\left[\frac{1}{r}\frac{\partial^2}{\partial t\partial y_j}\left(\rho v_j\right)\right]-\frac{\partial}{\partial y_j}\left[\frac{1}{r}\frac{\partial}{\partial t}\left(\rho v_j\right)\right]$$

$$=\left[-\frac{1}{r}\frac{\partial^2}{\partial t^2}\rho'\right]-\frac{\partial}{\partial y_j}\left[\frac{1}{r}\frac{\partial}{\partial t}\left(\rho v_j\right)\right]$$

- After using the equation of mass conservation. Hence it follows that

$$\frac{\partial^2}{\partial x_i\partial x_j}\left[\frac{T_{ij}}{r}\right]=-\frac{\partial^2}{\partial y_i\partial x_j}\left[\frac{\rho v_i v_j+p_{ij}}{r}\right]-\frac{\partial}{\partial y_j}\left[\frac{1}{r}\frac{\partial}{\partial t}\left(\rho v_j\right)\right]$$

- Then, integrate over a stationary volume,

$$\frac{\partial^2}{\partial x_i\partial x_j}\int_V\left[\frac{T_{ij}}{r}\right]dV=-\frac{\partial}{\partial x_j}\int_S\left[\frac{\rho v_i v_j+p_{ij}}{r}\right]dS-\frac{\partial}{\partial t}\int_S\left[\left(\frac{\rho\mathbf{v}\cdot\mathbf{n}}{r}\right)\right]dS$$

# Sources of Sound

## ❖ The sound from the stationary foreign bodies

- The result allows Lighthill’s acoustic analogy

$$\rho'(x,t) = \frac{1}{4\pi c^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_V \left[\frac{T_{ij}}{r}\right] dV \qquad \text{…..quadrupole}$$

$$-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_j}\int_S n_i \left[\frac{\rho v_i v_j + p_{ij}}{r}\right] dS \qquad \text{…..dipole}$$

$$+\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int_S \left[\frac{\rho \mathbf{v}\cdot\mathbf{n}}{r}\right] dS \qquad \text{…..monopole}$$

- Note
  - 1) The quadrupole field interior to the body = monopole + dipole

$$\frac{\partial^2}{\partial x_i \partial x_j}\int_{V_0} \left[\frac{T_{ij}}{r}\right] dV = -\frac{\partial}{\partial x_j}\int_S n_i \left[\frac{\rho v_i v_j + p_{ij}}{r}\right] dS$$

$$+\frac{\partial}{\partial t}\int_S \left[\frac{\rho \mathbf{v}\cdot\mathbf{n}}{r}\right] dS$$

# Sources of Sound

## ❖ The sound from the stationary foreign bodies

- 2) If the foreign bodies are motionless and rigid (***n·v**=0*)

$$-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_i}\int_s n_i\left[\frac{p}{r}\right]ds$$

- in compact body, $-\frac{1}{4\pi c^2}\frac{\partial}{\partial x_i}\left[\frac{F_i}{r}\right]$ $\left(F_i=\int_S n_i p'dS\right)$

F$_i$ is "Forces applied" to the body => Dipole sources

$$F \approx \rho_0 U^2 D^2 \qquad \rho' \approx \rho_0 M^3 \frac{D}{r}$$

$$\frac{\partial}{\partial x_i} \approx \frac{M}{D}\left(\approx \frac{1}{\lambda}\right) \qquad \overline{\rho'^2} \approx \rho_0^2 M^6 \frac{D^2}{r^2}$$

- Body forces are more efficient to generate sound by a factor of $M^{-2}$ than low Mach number free turbulence

# The sound from the stationary foreign bodies

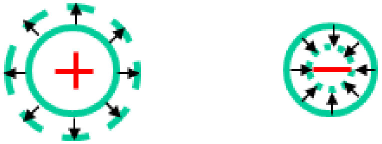
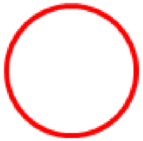
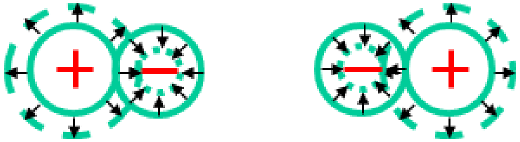
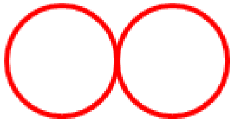
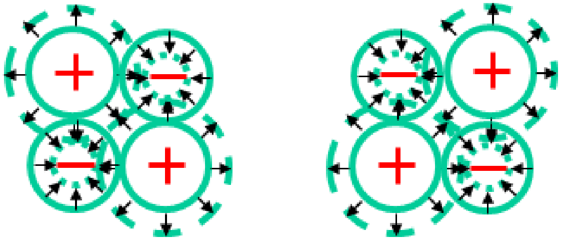
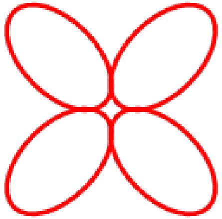
- 3) monopole source

$$-\frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left[\frac{Q}{r}\right] \quad \left(Q = \rho_0 \int_S v_n dS\right)$$

$$\rho' \approx \rho_0 M^2 \frac{D}{r} \quad \text{and} \quad \overline{\rho'^2} \approx \rho_0^2 M^4 \frac{D^2}{r^2}$$

- More effective than “dipole”

# Sources of Sound

| *Source type* | *Radiation characteristics* | *Radiation Pattern* | *Radiated Acoustic Energy* |
|---|---|---|---|
| Monopole |  |  | $\rho L^2 \frac{U^4}{c}$ |
| Dipole |  |  | $\rho L^2 \frac{U^6}{c^3}$ |
| Quadrupole |  |  | $\rho L^2 \frac{U^8}{c^5}$ |

$$\frac{U^2}{c^2} = M^2$$

$$\frac{U^2}{c^2} = M^2$$